

Vacuum polarization of the quantized massive scalar field in Reissner-Nordström spacetime

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The approximation of the renormalized stress-energy tensor of the quantized massive scalar field in Reissner-Nordström spacetime is constructed. It is achieved by functional differentiation of the first two nonvanishing terms of the Schwinger-DeWitt expansion involving the coincidence limit of the Hadamard-Minakshisundaram-DeWitt-Seely coefficients $[a_3]$ and $[a_4]$ with respect to the metric tensor. It is shown, by comparison with the existing numerical results, that inclusion of the second-order term leads to substantial improvement of the approximation of the exact stress-energy tensor. The approximation to the field fluctuation, $\langle \phi^2 \rangle$, is constructed and briefly discussed.

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I. INTRODUCTION

Recently, there has been a renewal of interest in calculations of the coincidence limits of the covariant derivatives of various bitensors and bispinors, such as the world function $\sigma(x, x')$, van Vleck-Morette determinant, bivectors and bispinors of the parallel displacement and the objects constructed from them, with the special emphasis put on the diagonal Hadamard-Minakshisundaram-DeWitt-Seely (HMDS) coefficients $[a_n(x, x')]$ and the renormalized stress-energy tensor [1–5]. This has largely been stimulated by new methods of computations as well as improvement of the computer algebra and the recent findings substantially extended previous results [6–15].

In this article we construct the approximation to the renormalized stress-energy tensor of the massive scalar field with arbitrary curvature coupling in the spacetime of the Reissner-Nordström black hole. This tensor has been the subject of previous studies both in the massless and massive cases. The extensive numerical calculations have been reported in Ref. [16], where, additionally, the first-order approximation in the large mass limit (which consists of the terms proportional to m^{-2}) has been developed. These calculations were based on the sixth-order WKB approximation of the solutions of the scalar field equations and the summation of the thus obtained mode functions. The latter result has been reconstructed with the aid of the Schwinger-DeWitt approximation of the renormalized effective action, W_R , and subsequently generalized to the spinor and vector fields in the $R = 0$ geometries in Ref. [1]. The approximate stress-energy tensor in a general background geometry has been constructed in Ref. [2]. A careful analysis carried out in Ref. [17] demonstrated that the approximation is reasonable for $Mm > 2$ and should become increasingly accurate as the ratio λ_C/\mathcal{L} decreases, leaving, however, room for further improvement. Here M is the mass of the Reissner-Nordström black hole, λ_C is the Compton length associated with the massive field and \mathcal{L} is the characteristic length of the background geometry.

Our aim is to provide a better approximation than those proposed in Refs. [1, 2, 16] and this is achieved by the inclusion of all relevant terms of the background dimensionality 8 (which equals twice the order of the coefficient $[a_n]$ or the total number of derivatives of the metric tensor in each term) in the renormalized effective action. Such terms are proportional to m^{-4} and constitute the next-to-leading order of the approximation. The approximate stress-energy tensor can be obtained by the functional differentiation of the thus constructed effective action with respect to the metric tensor.

The basic building blocks of the renormalized one-loop W_R are the coincidence limits of the HMDS coefficients which are local quantities constructed from the Riemann tensor, its covariant derivatives and contractions. The spin of the field and the type of the curvature coupling enters W_R through the numerical coefficients and the background dimensionality of $[a_3]$ and $[a_4]$ is 6 and 8, respectively. In general, the coefficient $[a_n]$ is a linear combination of the Riemann monomials and belongs to $\bigoplus_{q=1}^n \mathcal{R}_{2n,q}^0$, where $\mathcal{R}_{s,q}^r$ is a vector space of Riemannian polynomials of rank r (the number of free tensor indices), degree q (number of factors) and order s (number of derivatives).

The calculation of the functional derivatives of the effective action with respect to g_{ab} is rather tedious and time consuming process. Fortunately, for the spherically-symmetric geometries one can considerably simplify calculations. Indeed, substituting the line element of the general static and spherically-symmetric spacetime expressed in the Schwarzschild gauge into the effective action and performing simple integrations one obtains a reduced functional that depends on the two metric potentials. Two components of the stress-energy tensor are given by appropriate Lagrange derivatives of the Lagrangian of the reduced action functional with respect to the time and radial components of the metric tensor. This approach is justified by the symmetric criticality theorems of Palais [18, 19] and the remaining

components can easily be calculated from the covariant conservation equation.

The coefficients $[a_n]$ are also the basic building blocks of the approximate field fluctuation $\langle\phi^2\rangle$ and the knowledge of $[a_2]$, $[a_3]$ and $[a_4]$ allows for detailed analysis of the role played by the next-to-leading and the next-to-next-to-leading terms. One expects that some general features exhibited by $\langle\phi^2\rangle$ are also shared by the stress-energy tensor and if so it would be a fortunate circumstance.

The paper is organized as follows. In Sec. II the basic building blocks of the approximation, i.e., the coincidence limits of the HMDS coefficients $[a_3]$ and $[a_4]$ as calculated within the framework of the covariant DeWitt method are presented in maximally condensed form. The renormalized expectation value $\langle\phi^2\rangle$ in the Reissner-Nordström geometry is computed in Sec. III. The effective action and the stress-energy tensor of the quantized massive scalar fields in the Reissner-Nordström geometry is constructed and discussed in Sec. IV. A comparison with the numeric results indicate that inclusion of the next-to-leading term substantially improves approximation.

II. HADAMARD-MINAKSHISUNDARAM-DEWITT-SEELY COEFFICIENTS $[a_3]$ AND $[a_4]$.

It is a well-known fact that for sufficiently massive quantized fields, i.e., when the Compton length λ_C is smaller than the characteristic radius of curvature of the background geometry, the asymptotic expansion of the effective action in powers of m^{-2} may be used to describe various physical phenomena. It is because the nonlocal contribution to the effective action can be neglected whereas the vacuum polarization part is determined by the local geometry. The renormalized effective action constructed within the framework of the Schwinger-DeWitt approximation for the quantized massive scalar field satisfying the covariant Klein-Gordon equation with the curvature coupling ξ

$$(\square - \xi R - m^2)\phi = 0, \quad (1)$$

can be written in the form

$$W_R = \frac{1}{32\pi^2} \sum_{n=3}^{\infty} \frac{(n-3)!}{(m^2)^{n-2}} \int d^4x \sqrt{g} [a_n], \quad (2)$$

where $[a_n]$ is constructed from the Riemann tensor, its covariant derivatives up to $2n-2$ order and appropriate contractions. For the technical details of this approach the reader is referred, for example, to Refs. [20, 21] and the references cited therein.

Inspection of Eq. (2) shows that the lowest term of the approximate W_R is to be constructed from the (integrated) coincidence limit of the fourth HMDS coefficient, $[a_3]$, whereas the next to leading term is constructed from $[a_4]$. Here we will confine ourselves to the first two terms of the expansion (2).

The coefficients $a_n(x, x')$ satisfy the equation

$$\sigma^{;i} a_{n;i} + n a_n - \Delta^{-1/2} \square \left(\Delta^{1/2} a_{n-1} \right) + \xi R a_{n-1} = 0, \quad (3)$$

with the boundary condition $a_0(x, x') = 1$. Here $\Delta(x, x')$ is the van Vleck-Morette determinant and the biscalar $\sigma(x, x')$ is equal to one half the square of the distance along the geodesic between x and x' . From the recursive relation (3) it is clear that to construct, say, $[a_4]$, one has to calculate coincidence limits of a_3 , $a_{3;i_1}$ and $a_{3;i_1 i_2}$, which, in turn, require calculation of $[a_2]$ to $[a_{2;i_1 \dots i_4}]$, and so forth. Both $[a_3]$ and $[a_4]$ used in this paper have been calculated within the framework of the manifestly covariant method proposed by DeWitt with the aid of the FORM and its multithread version TFORM [22, 23]. Further simplifications, after appropriate syntax conversion, were carried out with the aid of the package INVAR [24, 25].

Since the coefficients $[a_3]$ and $[a_4]$ are the basis of the present calculations we shall display them at length. The coefficient $[a_3]$, when simplified with the aid of the INVAR, can be written in the form

$$[a_3] = a_3^{(0)} + \xi a_3^{(1)} + \xi^2 a_3^{(2)} + \xi^3 a_3^{(3)}, \quad (4)$$

where

$$\begin{aligned} a_3^{(0)} = & \frac{11}{1680} R^3 + \frac{17}{5040} R_{;a} R^{;a} - \frac{1}{2520} R_{ab;c} R^{ab;c} - \frac{1}{1260} R_{ab;c} R^{ac;b} \\ & + \frac{1}{560} R_{abcd;e} R^{abcd;e} + \frac{1}{180} R R_{;a}{}^a + \frac{1}{280} R_{;a}{}^a{}_{;b}{}^b + \frac{1}{420} R_{;ab} R^{ab} \\ & - \frac{1}{630} R_{ab;c}{}^c R^{ab} - \frac{109}{2520} R R_{ab} R^{ab} + \frac{73}{1890} R_{ab} R_c{}^a R^{bc} + \frac{1}{210} R R_{abcd} R^{abcd} \\ & + \frac{1}{105} R_{ab;cd} R^{acbd} + \frac{19}{630} R_{ab} R_{cd} R^{acbd} - \frac{1}{189} R_{abcd} R_{ef}{}^{ab} R^{cdef}, \end{aligned} \quad (5)$$

$$\begin{aligned}
a_3^{(1)} = & -\frac{1}{72}R^3 - \frac{1}{30}R_{;a}R^{;a} - \frac{11}{180}RR_{;a}{}^a - \frac{1}{180}RR_{abcd}R^{abcd} \\
& - \frac{1}{60}R_{;a}{}^a{}_{;b}{}^b - \frac{1}{90}R_{;ab}R^{ab} + \frac{1}{180}RR_{ab}R^{ab},
\end{aligned} \tag{6}$$

$$a_3^{(2)} = \frac{1}{12}R^3 + \frac{1}{12}R_{;a}R^{;a} + \frac{1}{6}RR_{;a}{}^a \tag{7}$$

and

$$a_3^{(3)} = -\frac{1}{6}R^3. \tag{8}$$

Before we proceed to the coefficient $[a_4]$ let us discuss briefly this result. The package INVAR tries to expand each Riemann monomial in the basis of the independent Riemann invariants with no free indices ($r = 0$). This is achieved by defining polynomial relations between dependent monomials and the basis. Additionally, a number of dimensionally dependent identities have been implemented. All this is of great practical importance since it is common that small changes in the computational strategy can yield great differences in the theoretically equivalent results and the explicit demonstration of their equality is extremely difficult. Having at one's disposal a basis and rules provided by INVAR one can easily establish equivalence of the results by construction a unique set of numerical coefficients and our calculations of $[a_3]$ and $[a_4]$ have been compared and checked this way. It should be noted that the number of terms in $[a_3]$ has been reduced as compared to the result presented, e.g., in Ref. [26], but, of course, they are identical up to relatively simple identities satisfied by the Riemann tensor.

The coefficient $[a_4]$ is, on the other hand, extremely complicated and even after massive simplifications it consists of 113 terms of dimensionality of $[1/length]^8$. The coefficient $[a_4]$ can be written in the form

$$[a_4] = a_4^{(0)} + \xi a_4^{(1)} + \xi^2 a_4^{(2)} + \xi^3 a_4^{(3)} + \xi^4 a_4^{(4)}, \tag{9}$$

where

$$\begin{aligned}
a_4^{(0)} = & \frac{1}{3780}R_{;a}{}^a{}_{;b}{}^b{}_{;c}{}^c + \frac{5743}{1814400}R^4 + \frac{229}{30240}RR_{;a}R^{;a} - \frac{241}{15120}RR_{ab;c}R^{ab;c} - \frac{1}{840}RR_{ab;c}R^{ac;b} \\
& + \frac{1}{672}RR_{abcd;e}R^{abcd;e} + \frac{1}{1800}R_{;a}{}^a{}_{;b}{}^b + \frac{13}{25200}R_{;ab}R^{;ab} - \frac{1}{405}RR_{abcd}R_{ef}{}^{ab}R^{cdef} \\
& + \frac{13}{37800}R_{;ab}R^{ab}{}_{;c}{}^c - \frac{1}{8400}R_{ab;c}{}^cR^{ab}{}_{;d}{}^d + \frac{13}{12600}R_{ab;cd}R^{ab;cd} - \frac{1}{450}R_{ab;cd}R^{ac;bd} \\
& + \frac{13}{12600}R_{ab;cd}R^{cd;ab} + \frac{1}{3150}R_{abcd;ef}R^{abcd;ef} + \frac{11}{7560}R_{;a}R_{;b}{}^{ab} + \frac{1}{1890}R_{ab;c}R^{;abc} \\
& - \frac{1}{7560}R_{ab;c}R^{ab;c}{}_{;d}{}^d - \frac{1}{3780}R_{ab;c}R^{ac;b}{}_{;d}{}^d + \frac{1}{315}R_{abcd;e}R^{ac;bde} + \frac{13}{9450}R_{abcd}R_{efg}{}^bR_h{}^{gcd}R^{ahef} \\
& - \frac{13}{15120}R_{;a}R_{;b}R^{ab} + \frac{1}{4200}RR_{;ab}R^{ab} - \frac{5}{504}RR_{ab;c}R^{ab} + \frac{29}{75600}R_{abcd}R_{efgh}R^{abcd}R^{efgh} \\
& + \frac{1}{1890}R_{;abc}{}^cR^{ab} - \frac{1}{7560}R_{ab;c}{}^c{}_{;d}{}^dR^{ab} - \frac{7253}{302400}R^2R_{ab}R^{ab} - \frac{271}{7560}R_{;a}R_{bc}{}^{;a}R^{bc} + \frac{4}{1575}R^2R_{;a}{}^a \\
& + \frac{1}{630}R_{;a}R_{b;c}{}^aR^{bc} - \frac{17}{3150}R_{;a}{}^aR_{bc}R^{bc} + \frac{22}{675}RR_{ab}R_c{}^aR^{bc} + \frac{89}{1890}R_{ab;c}R_d{}^{a;c}R^{bd} \\
& + \frac{1}{378}R_{ab;c}R_d{}^{c;a}R^{bd} + \frac{83}{6300}R_{ab;c}{}^cR_d{}^aR^{bd} - \frac{29}{4725}R_{ab;cd}R^{ac}R^{bd} - \frac{341}{18900}R_{ab}R_{cd}R^{ac}R^{bd} \\
& - \frac{4}{945}R_{ab;c}R_d{}^{a;b}R^{cd} + \frac{1}{1260}R_{ab;c}R^{ab}{}_{;d}{}^dR^{cd} + \frac{109}{18900}R_{ab;cd}R^{ab}R^{cd} + \frac{703}{151200}R_{ab}R_{cd}R^{ab}R^{cd} \\
& + \frac{71}{3780}R_{ab;c}R_d{}^a{}_{;e}{}^b{}_{;e}{}^cR^{de} + \frac{1}{126}R_{ab;c}R_d{}^a{}_{;e}{}^c{}_{;e}{}^bR^{de} - \frac{1}{140}R_{abcd;e}R_f{}^{ace;b}R^{df} - \frac{1}{945}R_{;a}R_{bc;d}R^{abcd} \\
& + \frac{67}{33600}R^2R_{abcd}R^{abcd} + \frac{29}{6300}RR_{ab;cd}R^{acbd} + \frac{1}{945}R_{ab;cde}{}^eR^{acbd} + \frac{17}{18900}R_{;ab}R_{cd}R^{acbd} \\
& + \frac{83}{10800}RR_{ab}R_{cd}R^{acbd} + \frac{29}{1890}R_{ab;c}R_{de}{}^{;c}R^{adbe} + \frac{32}{4725}R_{ab;c}{}^cR_{de}R^{adbe} - \frac{22}{4725}R_{ab;cd}R_e{}^bR^{adce}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4725} R_{ab;cd} R_{ef}^c R^{abef} - \frac{4}{945} R_{ab;c} R_{de}^b R^{aecd} + \frac{2}{1575} R_{abcd;ef} R^{bd} R^{aecf} + \frac{1}{1680} R R_{;a}^a R^b_b \\
& - \frac{1}{945} R_{ab;c} R_d^a R^{bcde} + \frac{29}{7560} R_{;a} R_{bcde}^a R^{bcde} + \frac{1}{1400} R_{;a}^a R_{bcde} R^{bcde} - \frac{73}{9450} R_{ab;cd} R_{ef}^{ad} R^{bcef} \\
& + \frac{1}{10800} R_{ab} R_{cd} R_{ef}^{ac} R^{bdef} + \frac{1}{315} R_{ab;c} R_{def}^{c;a} R^{bfde} - \frac{547}{302400} R_{ab} R^{ab} R_{cdef} R^{cdef} \\
& - \frac{1}{168} R_{abcd;e} R_{fg}^{ab;e} R^{cdfg} + \frac{13}{18900} R_{abcd} R_{efgh} R^{abef} R^{cdgh} + \frac{103}{12600} R_{ab} R_{cd} R_{ef}^a R^{cfde}, \tag{10}
\end{aligned}$$

$$\begin{aligned}
a_4^{(1)} = & -\frac{11}{1680} R^4 - \frac{1}{112} R R_{;a} R^a + \frac{1}{2520} R R_{ab;c} R^{ab;c} + \frac{1}{189} R R_{abcd} R_{ef}^{ab} R^{cdef} \\
& + \frac{1}{1260} R R_{ab;c} R^{ac;b} - \frac{1}{560} R R_{abcd;e} R^{abcd;e} - \frac{41}{5040} R^2 R_{;a}^a - \frac{1}{180} R_{;a}^a R_{;b}^b \\
& - \frac{1}{210} R_{;ab} R^{ab} - \frac{1}{630} R_{;ab} R^{ab;c} - \frac{1}{72} R_{;a} R_{;b}^{ab} - \frac{1}{420} R_{ab;c} R^{abc} - \frac{1}{840} R_{;a}^a R_{bcde} R^{bcde} \\
& - \frac{2}{315} R R_{;a}^a R^b_b - \frac{1}{840} R_{;a}^a R_{;b}^b R^c_c + \frac{17}{2520} R_{;a} R_{;b} R^{ab} - \frac{1}{315} R R_{;ab} R^{ab} - \frac{1}{420} R_{;a} R_{bcde}^a R^{bcde} \\
& + \frac{1}{630} R R_{ab;c} R^{ab} - \frac{1}{420} R_{;abc} R^{ab} + \frac{109}{2520} R^2 R_{ab} R^{ab} + \frac{1}{1260} R_{;a} R_{bc}^a R^{bc} \\
& + \frac{1}{630} R_{;a} R_b^a R^{bc} + \frac{1}{504} R_{;a}^a R_{bc} R^{bc} - \frac{73}{1890} R R_{ab} R_c^a R^{bc} + \frac{1}{630} R_{;a} R_{bc;d} R^{abcd} \\
& - \frac{1}{210} R^2 R_{abcd} R^{abcd} - \frac{1}{105} R R_{ab;cd} R^{abcd} - \frac{1}{210} R_{;ab} R_{cd} R^{abcd} - \frac{19}{630} R R_{ab} R_{cd} R^{abcd}, \tag{11}
\end{aligned}$$

$$\begin{aligned}
a_4^{(2)} = & \frac{1}{144} R^4 + \frac{17}{360} R R_{;a} R^a + \frac{2}{45} R^2 R_{;a}^a + \frac{1}{72} R_{;a}^a R_{;b}^b \\
& + \frac{1}{90} R_{;ab} R^{ab} + \frac{1}{30} R_{;a} R_{;b}^{ab} + \frac{1}{60} R R_{;a}^a R^b_b - \frac{1}{60} R_{;a} R_{;b} R^{ab} \\
& + \frac{1}{90} R R_{;ab} R^{ab} - \frac{1}{360} R^2 R_{ab} R^{ab} + \frac{1}{360} R^2 R_{abcd} R^{abcd}, \tag{12}
\end{aligned}$$

$$a_4^{(3)} = -\frac{1}{36} R^4 - \frac{1}{12} R R_{;a} R^a - \frac{1}{12} R^2 R_{;a}^a \tag{13}$$

and

$$a_4^{(4)} = \frac{1}{24} R^4. \tag{14}$$

Although the total divergences can be discarded when working with the effective action we have retained all the terms both in $[a_3]$ and $[a_4]$, simply because they are interesting in their own right and can be used in further calculations of, for example, the field fluctuation, $\langle \phi^2 \rangle$.

III. FIELD FLUCTUATION

Before proceeding to the calculation of the renormalized stress-energy tensor in the Reissner-Nordström geometry let us construct $\langle \phi^2 \rangle$ as it shares some of the general features of the full stress-energy tensor while simultaneously being computationally less involved. For example, if the next-to-leading term in the expansion leads to substantial improvement of the result it is likely that the quality of the analytic approximation of the stress-energy tensor would also improve. Similarly, if the vacuum polarization diverges on the event horizon it is quite probable that the stress-energy tensor is also divergent there.

The vacuum polarization has been studied by a number of authors both in the massive and massless cases. (See, e.g. [27, 28] and the references cited therein). Generally, within the Schwinger-DeWitt framework the field fluctuation can be expressed in terms of the coincidence limit of the coefficients $[a_n]$ and is given by

$$\langle \phi^2 \rangle_k = \frac{1}{16\pi^2} \sum_{n=2}^k \frac{(n-2)!}{m^{2(n-1)}} [a_n], \tag{15}$$

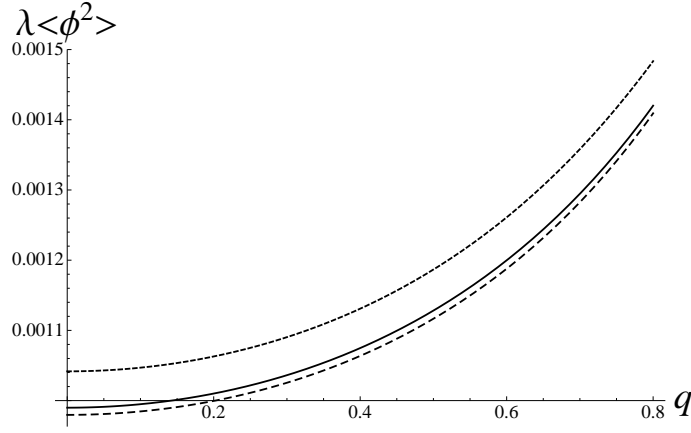


FIG. 1: This graph shows the rescaled values of the vacuum polarization $\langle\phi^2\rangle_i$ [$\lambda = 16\pi^2 M^2$] at the event horizon for the massive scalar field with $mM = 2$. The solid line corresponds to $\langle\phi^2\rangle_4$, the dashed line corresponds to $\langle\phi^2\rangle_3$ and the dotted line corresponds to $\langle\phi^2\rangle_2$.

where $k-1$ is the number of terms retained in the expansion. The leading term of the expansion is to be constructed from the coefficient $[a_2]$, which, for the scalar field satisfying Eq. (1) is given by

$$[a_2] = \frac{1}{180} R_{abcd} R^{abcd} - \frac{1}{180} R_{ab} R^{ab} + \frac{1}{6} \left(\frac{1}{5} - \xi \right) R_{;a}^a + \frac{1}{2} \left(\frac{1}{6} - \xi \right)^2 R^2. \quad (16)$$

Having at our disposal compact expressions describing the first three coefficients of the expansion (15) we can analyze the influence of the higher order terms on the final result. Routine calculations carried out in the Reissner-Nordström geometry give

$$[a_2] = \frac{1}{45r^6} \left(12M^2 - 24 \frac{MQ^2}{r} + 13 \frac{Q^4}{r^2} \right), \quad (17)$$

$$[a_3] = \frac{1}{r^8} \left(-\frac{194}{63} \frac{M^3}{r} + \frac{9}{7} M^2 + \frac{1111}{105} \frac{M^2 Q^2}{r^2} - \frac{132}{35} \frac{MQ^2}{r} - \frac{3644}{315} \frac{MQ^4}{r^3} + \frac{908}{315} \frac{Q^4}{r^2} + \frac{1156}{315} \frac{Q^6}{r^4} \right) \quad (18)$$

and

$$[a_4] = \frac{1}{r^{10}} \left(\frac{16549}{315} \frac{Q^8}{r^6} + \frac{50132}{675} \frac{Q^6}{r^4} - \frac{373652}{1575} \frac{Q^6 M}{r^5} + \frac{571027}{1575} \frac{Q^4 M^2}{r^4} + \frac{612}{25} \frac{Q^4}{r^2} - \frac{318476}{1575} \frac{MQ^4}{r^3} \right. \\ \left. + \frac{3757}{25} \frac{M^2 Q^2}{r^2} - \frac{113522}{525} \frac{M^3 Q^2}{r^3} - \frac{4352}{175} \frac{MQ^2}{r} + \frac{32}{5} M^2 + \frac{23264}{525} \frac{M^4}{r^2} - \frac{240}{7} \frac{M^3}{r} \right). \quad (19)$$

Now, in order to gain insight into the nature of the thus constructed approximations let us compare $\langle\phi^2\rangle_2$ (the leading term), $\langle\phi^2\rangle_3$ ($\langle\phi^2\rangle_2$ plus the next-to-leading term) and $\langle\phi^2\rangle_4$ ($\langle\phi^2\rangle_3$ plus the next-to-next-to-leading term). In Figs. 1 and 2 $\langle\phi^2\rangle_k$ for $k = 2, 3, 4$ at the event horizon is plotted against the admissible values of $|Q|/M$. It is seen that the second-order term considerably modifies the main approximation. On the other hand, however, $\langle\phi^2\rangle_3$ and $\langle\phi^2\rangle_4$ differ only slightly and consequently the changes in the field fluctuation caused by $[a_4]$ are relatively small. Similarly, for a given Q the modification caused by the next-to-next-to-leading term is small everywhere outside the event horizon. Therefore, it is reasonable to retain only the first two terms of the expansion (15). In what follows we shall demonstrate that a similar pattern holds for the stress-energy tensor and the first two terms of the expansion (2) provide a good approximation. There is, however, a profound difference between the two objects: to evaluate the renormalized stress-energy tensor one has to retain the terms constructed from $[a_3]$ and $[a_4]$ whereas analogous calculations of the field fluctuation require $[a_2]$ and $[a_3]$.

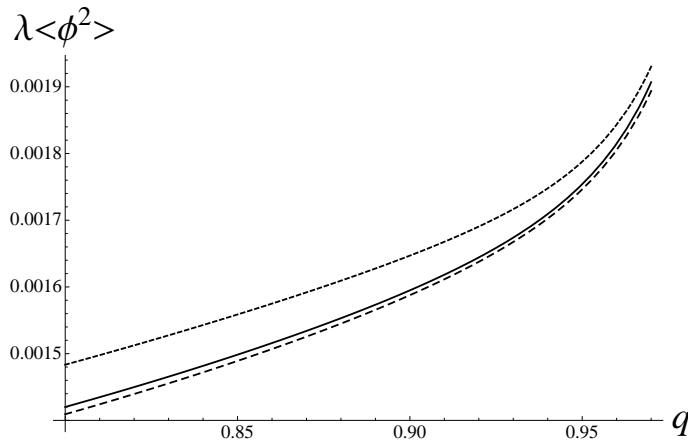


FIG. 2: This graph shows the rescaled values of the vacuum polarization $\langle\phi^2\rangle_i$ [$\lambda = 16\pi^2 M^2$] at the event horizon for the massive scalar field with $mM = 2$. The solid line corresponds to $\langle\phi^2\rangle_4$ the dashed line corresponds to $\langle\phi^2\rangle_3$ and the dotted line corresponds to $\langle\phi^2\rangle_2$.

IV. THE APPROXIMATE STRESS-ENERGY TENSOR

A. Effective action

Having at one's disposal the approximate effective action, W_R , the stress-energy tensor can be obtained from the standard formula

$$T^{ab} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{ab}} W_R. \quad (20)$$

The total action that leads to the semiclassical Einstein field equations can be written in the form

$$S_{total} = \frac{1}{16\pi} \int R g^{1/2} d^4x + S_m + W_R, \quad (21)$$

where S_m is the action of the classical sources and

$$W_R = \frac{1}{32\pi^2 m^2} \int [a_3] g^{1/2} d^4x + \frac{1}{32\pi^2 m^4} \int [a_4] g^{1/2} d^4x. \quad (22)$$

Now, in order to calculate the functional derivatives of the action with respect to the metric tensor the maximally simplified coefficients $[a_3(x, x')]$ and $[a_4(x, x')]$ have once again been converted into the FORM language. This allows us to make use of our powerful FORM codes for functional derivatives of the effective action. The final (mildly simplified) result for the renormalized stress-energy tensor of the massive scalar field with an arbitrary curvature coupling consists of a few thousand rather complicated terms and is stored in the FORM format. It will not be displayed here because it is too bulky. All our calculations in various geometries are always carried out using this very tensor and the results are compared (if necessary) with the analogous results constructed using different (possibly simpler) computational strategies.

For the spherically symmetric line element expressed in the Schwarzschild gauge

$$ds^2 = f^{(0)}(r) dt^2 + h^{(0)}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (23)$$

one can save a lot of work by using the reduced action functionals. Indeed, since the coefficients $[a_n]$ are constructed from various curvature invariants they depend solely on the functions $f^{(0)}(r)$ and $h^{(0)}(r)$, their radial derivatives and the radial coordinate. Therefore, one can easily perform simple integrations and reduce the problem to variations with respect to the functions $f^{(0)}(r)$ and $h^{(0)}(r)$.

The reduced action functional of the quantum part of the total action is

$$W_R^{reduced} = \frac{1}{32\pi^2 m^2} \int dr [a_3] (f^{(0)} h^{(0)})^{1/2} r^2 + \frac{1}{32\pi^2 m^4} \int dr [a_4] (f^{(0)} h^{(0)})^{1/2} r^2. \quad (24)$$

Although still tedious, this method requires substantially smaller number of operations than the general one. This procedure yields T^{tt} and T^{rr} components of the renormalized stress-energy tensor; the third algebraically independent component, $T_\theta^\theta = T_\phi^\phi$, can be obtained from the covariant conservation equation.

The quantum part of the total Lagrangian can schematically be written as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \quad (25)$$

where

$$\mathcal{L}_1 = \mathcal{L}_1 \left(f^{(0)}(r), \dots, f^{(6)}(r), h^{(0)}(r), \dots, h^{(5)}(r), r \right) \quad (26)$$

and

$$\mathcal{L}_2 = \mathcal{L}_2 \left(f^{(0)}(r), \dots, f^{(8)}(r), h^{(0)}(r), \dots, h^{(7)}(r), r \right). \quad (27)$$

$f^{(k)}$ and $h^{(k)}$ denote a k -th derivative of $f^{(0)}(r)$ and $h^{(0)}(r)$, respectively. Note that the coefficients $(32\pi^2 m^{2i})^{-1}$ have been absorbed into the definition of \mathcal{L}_i .

The full form of \mathcal{L}_1 and \mathcal{L}_2 calculated for the line element (24), when expanded, consists of 531 and 2157 terms, respectively, and will not be presented here for obvious reasons. In practice, there is no need to retain all the terms in Eqs. (4) and (9). For example, for the $R = 0$ class of metrics which is our main interest here all the terms in Eqs. (7), (8) and (12-14) do not contribute to the final result and the number of terms in $a_3^{(0)}$, $a_3^{(1)}$ as well as $a_4^{(0)}$ and $a_4^{(1)}$ is substantially reduced. Further simplifications can be obtained by neglecting the total divergences.

B. Approximate stress-energy tensor in Reissner-Nordström geometry

Now the stress-energy tensor can be obtained from the Euler-Lagrange equations

$$T_t^{(i)t} = 2 \left(\frac{f^{(0)}}{h^{(0)}} \right)^{1/2} \left[\frac{\partial}{\partial f^{(0)}} \mathcal{L}_i + \sum_{k=1}^{n(i)} (-1)^{k+1} \frac{d^k}{dr^k} \left(\frac{\partial}{\partial f^{(k)}} \mathcal{L}_i \right) \right] \quad (28)$$

and

$$T_r^{(i)r} = 2 \left(\frac{h^{(0)}}{f^{(0)}} \right)^{1/2} \left[\frac{\partial}{\partial h^{(0)}} \mathcal{L}_i + \sum_{k=1}^{s(i)} (-1)^{k+1} \frac{d^k}{dr^k} \left(\frac{\partial}{\partial h^{(k)}} \mathcal{L}_i \right) \right], \quad (29)$$

where $n(i)$ and $s(i)$ can easily be inferred from Eqs. (26) and (27). The angular components can be easily obtained from the covariant conservation equation $\nabla_a T^{ab} = 0$, which, for the line element (24), reduces to

$$T_\theta^{(i)\theta} = T_\phi^{(i)\phi} = -\frac{r}{4f^{(0)}} \left(T_t^{(i)t} - T_r^{(i)r} \right) \frac{d}{dr} f^{(0)} + \frac{r}{2} \frac{d}{dr} T_r^{(i)r} + T_r^{(i)r}, \quad (30)$$

where $i = 1, 2$.

Before we start calculations in the Reissner-Nordström geometry let us discuss some features of the approximation to the stress-energy tensor which can be deduced from the coefficients $[a_3]$ and $[a_4]$. First observe that the stress-energy tensor depends linearly on the coupling parameter ξ and can be written as a sum of tensors of the type

$$T_a^{(i)b} = \frac{1}{\pi^2 m^2 i r^{2(3+i)}} \left(C_a^{(i)b} + \eta D_a^{(i)b} \right), \quad (31)$$

where $\eta = \xi - 1/6$. Further, it should be noted that since T_a^b is constructed solely from the Riemann tensor and its covariant derivatives it is regular for regular metrics. This property is guaranteed by the polynomial character of the result and the factorization:

$$T_t^{(i)t} - T_r^{(i)r} = f^{(0)}(r) P^{(i)}(r), \quad (32)$$

where $P^{(i)}(r)$ are the regular functions.

Although the first-order approximation to the stress-energy tensor in the Reissner-Nordström geometry is known, we shall display it for reader's convenience. Making use of our general formulas, after some algebra, we conclude that the approximation has the form (32), where

$$C_t^{(1)t} = \frac{3}{112} Q^2 - \frac{13}{315} \frac{Q^6}{r^4} - \frac{19}{672} M^2 - \frac{4}{105} \frac{MQ^2}{r} + \frac{313}{5040} \frac{M^3}{r} - \frac{101}{15120} \frac{Q^4}{r^2} - \frac{769}{10080} \frac{M^2 Q^2}{r^2} + \frac{257}{2520} \frac{MQ^4}{r^3}, \quad (33)$$

$$D_t^{(1)t} = -\frac{11}{10} \frac{M^3}{r} - \frac{7}{5} \frac{MQ^2}{r} + \frac{217}{60} \frac{M^2 Q^2}{r^2} - \frac{113}{30} \frac{MQ^4}{r^3} + \frac{91}{90} \frac{Q^4}{r^2} + \frac{91}{80} \frac{Q^6}{r^4} + \frac{1}{2} M^2, \quad (34)$$

$$C_r^{(1)r} = \frac{1}{96} M^2 + \frac{3}{560} Q^2 + \frac{421}{15120} \frac{Q^4}{r^2} - \frac{31}{630} \frac{MQ^2}{r} - \frac{11}{720} \frac{M^3}{r} + \frac{37}{2520} \frac{Q^6}{r^4} + \frac{709}{10080} \frac{M^2 Q^2}{r^2} - \frac{23}{360} \frac{MQ^4}{r^3}, \quad (35)$$

$$D_r^{(1)r} = \frac{7}{15} \frac{MQ^2}{r} - \frac{49}{60} \frac{M^2 Q^2}{r^2} - \frac{13}{45} \frac{Q^4}{r^2} - \frac{1}{5} M^2 + \frac{7}{10} \frac{MQ^4}{r^3} - \frac{13}{80} \frac{Q^6}{r^4} + \frac{3}{10} \frac{M^3}{r}, \quad (36)$$

$$C_\theta^{(1)\theta} = C_\phi^{(1)\phi} = \frac{407}{2520} \frac{MQ^2}{r} - \frac{M^2}{32} + \frac{367}{5040} \frac{M^3}{r} + \frac{863}{2520} \frac{MQ^4}{r^3} - \frac{761}{7560} \frac{Q^4}{r^2} - \frac{367}{1120} \frac{M^2 Q^2}{r^2} - \frac{73}{720} \frac{Q^6}{r^4} - \frac{9Q^2}{560} \quad (37)$$

and

$$D_\theta^{(1)\theta} = D_\phi^{(1)\phi} = \frac{91}{20} \frac{M^2 Q^2}{r^2} - \frac{7}{5} \frac{M^3}{r} - \frac{49}{30} \frac{MQ^2}{r} - \frac{71}{15} \frac{MQ^4}{r^3} + \frac{52}{45} \frac{Q^4}{r^2} + \frac{3}{5} M^2 + \frac{117}{80} \frac{Q^6}{r^4}. \quad (38)$$

This tensor is identical to that constructed in Refs. [1, 16].

Now, let us return to the tensor $T_a^{(2)b}$. The calculations that lead to this object are far more complicated than the analogous calculations of the first-order term and require heavy use of the computer algebra. All these efforts however will pay off and give us substantially better approximation. In recent publications [4, 29] it has been argued that the minimal approximation constructed within the framework of the Schwinger-DeWitt method should consist of the two first terms of the expansion (2). This observation was based on the analyses carried out in the Schwarzschild and the Bertotti-Robinson geometries. Specifically, it has been demonstrated that the approximation of the stress-energy tensor in the Schwarzschild spacetime constructed from $[a_3]$ and $[a_4]$ is substantially better than the analogous approximation calculated from the coefficient $[a_3]$ alone and this by itself justifies the introduction of the second order term in that case. We shall show that similar behavior occurs in the spacetime of the Reissner-Nordström black hole. Moreover, the higher order terms may dramatically change the type of the solutions of the semiclassical Einstein field equations. An interesting example in this regard is given by the Bertotti-Robinson geometry [30, 31]. Specifically, it can be shown that although the Bertotti-Robinson geometry is a self-consistent solution of the semiclassical Einstein field equations with the source term given solely by the leading term of the renormalized stress-energy tensor [1, 32–34] it does not remain so when the next-to-leading term is taken into account. To guarantee that the Bertotti-Robinson spacetime is the solution of the semiclassical equations one has to introduce the (negative) cosmological constant. It should be noted that addition of the electric charge to the system does not change this behavior.

The second-order term has, as expected, the form (32), where

$$C_t^{(2)t} = -\frac{11}{40} M^2 + \frac{1}{12} Q^2 - \frac{255229}{151200} \frac{Q^8}{r^6} + \frac{2833}{2100} \frac{M^3}{r} + \frac{41063}{6300} \frac{MQ^4}{r^3} + \frac{157}{350} \frac{MQ^2}{r} - \frac{409}{525} \frac{Q^4}{r^2} - \frac{13583}{8400} \frac{M^4}{r^2} + \frac{95839}{12600} \frac{MQ^6}{r^5} - \frac{190013}{75600} \frac{Q^6}{r^4} - \frac{6131}{1400} \frac{M^2 Q^2}{r^2} - \frac{287009}{25200} \frac{M^2 Q^4}{r^4} + \frac{83611}{12600} \frac{M^3 Q^2}{r^3}, \quad (39)$$

$$D_t^{(2)t} = \frac{1649}{56} \frac{M^4}{r^2} + \frac{56361}{560} \frac{M^2 Q^2}{r^2} - \frac{2785}{21} \frac{MQ^4}{r^3} + \frac{9}{2} M^2 - \frac{47}{2} \frac{M^3}{r} + \frac{390577}{1680} \frac{M^2 Q^4}{r^4} - \frac{2251}{15} \frac{MQ^6}{r^5} - \frac{594}{35} \frac{MQ^2}{r} - \frac{71087}{504} \frac{M^3 Q^2}{r^3} + \frac{227}{14} \frac{Q^4}{r^2} + \frac{20207}{420} \frac{Q^6}{r^4} + \frac{41327}{1260} \frac{Q^8}{r^6}, \quad (40)$$

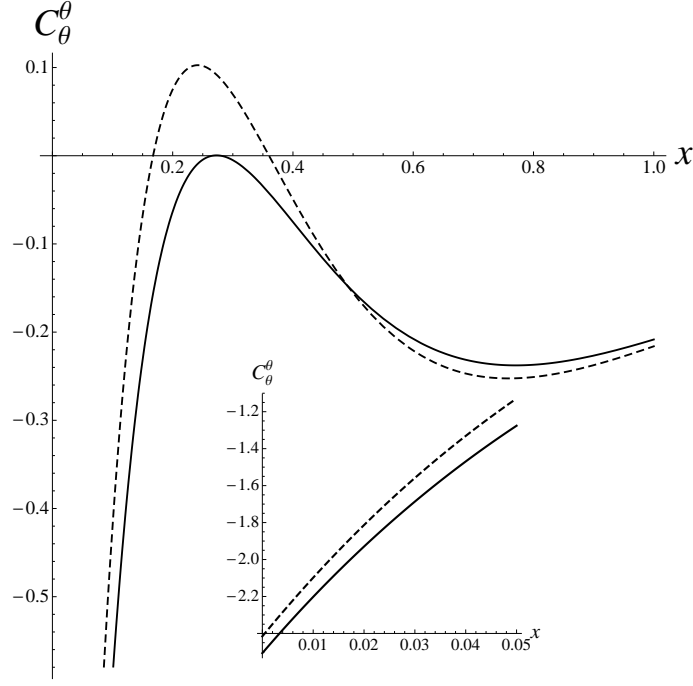


FIG. 3: This graph shows the rescaled values of C_θ^θ [$\lambda = 90(8M)^4\pi^2$] as function of $x = (r - r_+)/M$ for massive scalar field with $mM = 2$ and $|Q|/M = 0.95$. The solid line corresponds to the improved approximation whereas the dashed line to the first order Schwinger-DeWitt approximation. In the small panel the near horizon behavior of C_θ^θ is displayed

$$C_r^{(2)r} = \frac{3}{40} M^2 + \frac{1}{84} Q^2 + \frac{34463}{151200} \frac{Q^8}{r^6} - \frac{97}{300} \frac{M^3}{r} - \frac{7897}{4725} \frac{MQ^4}{r^3} - \frac{239}{700} \frac{MQ^2}{r} + \frac{247}{945} \frac{Q^4}{r^2} \\ + \frac{2753}{8400} \frac{M^4}{r^2} - \frac{151}{120} \frac{MQ^6}{r^5} + \frac{4321}{8400} \frac{Q^6}{r^4} + \frac{1531}{1050} \frac{M^2Q^2}{r^2} + \frac{11581}{5040} \frac{M^2Q^4}{r^4} - \frac{19907}{12600} \frac{M^3Q^2}{r^3}, \quad (41)$$

$$D_r^{(2)r} = -\frac{291}{56} \frac{M^4}{r^2} - \frac{34127}{1680} \frac{M^2Q^2}{r^2} + \frac{14911}{630} \frac{MQ^4}{r^3} - \frac{9}{7} M^2 + \frac{151}{28} \frac{M^3}{r} - \frac{3487}{112} \frac{M^2Q^4}{r^4} \\ + \frac{254}{15} \frac{MQ^6}{r^5} + \frac{297}{70} \frac{MQ^2}{r} + \frac{1567}{72} \frac{M^3Q^2}{r^3} - \frac{227}{63} \frac{Q^4}{r^2} - \frac{9433}{1260} \frac{Q^6}{r^4} - \frac{3757}{1260} \frac{Q^8}{r^6}, \quad (42)$$

$$C_\theta^{(2)\theta} = C_\phi^{(2)\phi} = -\frac{3}{10} M^2 - \frac{1}{21} Q^2 - \frac{386087}{151200} \frac{Q^8}{r^6} + \frac{163}{100} \frac{M^3}{r} + \frac{55159}{5400} \frac{MQ^4}{r^3} + \frac{2101}{1400} \frac{MQ^2}{r} - \frac{961}{756} \frac{Q^4}{r^2} \\ - \frac{17849}{8400} \frac{M^4}{r^2} + \frac{73417}{6300} \frac{MQ^6}{r^5} - \frac{137681}{37800} \frac{Q^6}{r^4} - \frac{16657}{2100} \frac{M^2Q^2}{r^2} - \frac{18259}{1008} \frac{M^2Q^4}{r^4} + \frac{138431}{12600} \frac{M^3Q^2}{r^3} \quad (43)$$

and

$$D_\theta^{(2)\theta} = D_\phi^{(2)\phi} = \frac{485}{14} \frac{M^4}{r^2} + \frac{38663}{336} \frac{M^2Q^2}{r^2} - \frac{189871}{1260} \frac{MQ^4}{r^3} + \frac{36}{7} M^2 - \frac{1521}{56} \frac{M^3}{r} + \frac{151701}{560} \frac{M^2Q^4}{r^4} \\ - \frac{18446}{105} \frac{MQ^6}{r^5} - \frac{2673}{140} \frac{MQ^2}{r} - \frac{41513}{252} \frac{M^3Q^2}{r^3} + \frac{1135}{63} \frac{Q^4}{r^2} + \frac{9869}{180} \frac{Q^6}{r^4} + \frac{48841}{1260} \frac{Q^8}{r^6}. \quad (44)$$

The angular components of the renormalized stress-energy tensor of the massive scalar field in the Reissner-Nordström geometry has been calculated numerically for $mM = 2$ and $|Q|/M = 0.95$. (Similar calculations have also been carried out for $|Q| = 0$.) However, in view of further applications, it is preferable to have at one's disposal simple and accurate general analytic formulas describing the functional dependence of the stress-energy tensor on the metric. Such

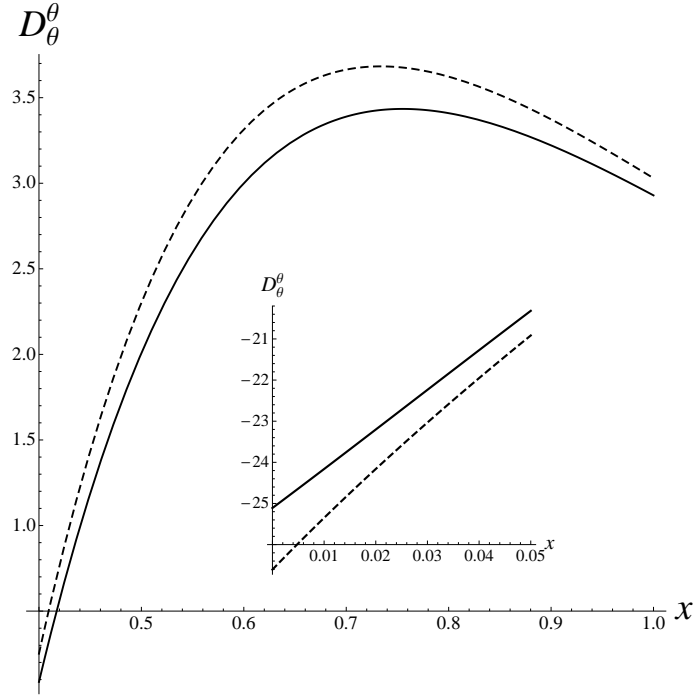


FIG. 4: This graph shows the rescaled values of D_θ^θ [$\lambda = 90(8M)^4\pi^2$] as function of $x = (r - r_+)/M$ for massive scalar field with $mM = 2$ and $|Q|/M = 0.95$. The solid line corresponds to the improved approximation whereas the dashed line to the first order Schwinger-DeWitt approximation. In the small panel the near horizon behavior of D_θ^θ is displayed.

general formulas can easily be applied to the concrete line element provided some general requirements concerning the geometry and the mass of the quantized field are satisfied. Consequently, it is of interest to compare the approximation constructed from $[a_3]$ and $[a_4]$ with the results of the numerical calculations of the conformal and nonconformal contribution to the total stress-energy tensor

$$T_\theta^\theta = C_\theta^\theta + \left(\xi - \frac{1}{6}\right) D_\theta^\theta. \quad (45)$$

as presented in ref. [16]

A comparison of Figs. 3 and 4 of the present paper with the Figs. 10 and 11 of Ref. [16] clearly shows that although the first order approximation correctly reproduces qualitative behavior of C_θ^θ and D_θ^θ , the inclusion of the next-to-leading term substantially improves the approximation of the stress-energy tensor even in the closest vicinity of the event horizon. One expects that this approximation is even better for $mM > 2$. A lesson that follows from this demonstration is that the next-to-leading term plays, or at least may play, an important role in the calculations and it can be ignored only after careful examination. Similar behavior of the next-to-leading term of the stress-energy tensor in the Schwarzschild spacetime has been reported in Ref. [4]. One expects therefore that this pattern holds for all values satisfying $0 \leq |Q|/M \leq 0.95$. There is also good reason to believe that it is true for all admissible values of q .

At the event horizon of the extreme Reissner-Nordström black hole the stress-energy tensor can be written in the remarkably simple form

$$T_a^{(1)b} = \frac{1}{\pi^2 m^2 M^6} \left(\frac{1}{3780} - \frac{\eta}{720} \right) \text{diag}[1, 1, -1, -1], \quad (46)$$

$$T_a^{(2)b} = -\frac{11}{151200\pi^2 m^4 M^8} \text{diag}[1, 1, 1, 1]. \quad (47)$$

As the geometry in the vicinity of the degenerate Reissner-Nordström solution is precisely that of the Bertotti-Robinson one can easily calculate the approximate stress-energy tensor. Indeed, due to homogeneity ($R_{abcd;e} = 0$), vanishing of the Ricci scalar and the Weyl tensor ($C_{abcd} = 0$), the stress-energy tensor can be expressed solely in

terms of the Ricci tensor. Making use of the general formulas the first order approximation to the stress-energy tensor can be written in the form

$$32\pi^2 m^2 T^{(1)ij} = -\frac{5-14\xi}{1260} R_{ab} R^{ab} R^{ij}, \quad (48)$$

whereas when the above conditions are satisfied the next-to-leading term reads

$$32\pi^2 m^2 T^{(2)ij} = -\frac{11}{75600} R_{ab} R^{ab} R_{cd} R^{cd} g^{ij}. \quad (49)$$

Simple calculation shows that (48) and (49) in the Bertotti-Robinson geometry are precisely equivalent to the stress-energy tensor at the degenerate horizon of the extreme Reissner-Nordström black hole and this may be regarded as the additional useful check of the calculations.

V. FINAL REMARKS

In this work our goal was to construct the approximate field fluctuation and renormalized stress-energy tensor of the quantized massive field in the spacetime of the Reissner-Nordström black hole and to investigate how the higher-order terms of the expansions (2) and (15) affect the final results. The general formulas describing the both quantities are extremely complex, but, fortunately, there are massive simplifications when applied to the static and spherically symmetric geometries. A comparison with the numeric calculations reported in a classic paper by Anderson, Hiscock and Samuel [16] shows that the next-to-leading term substantially improves the approximation. It has been found that in both cases the minimal approximations are to be constructed from the first two terms of (15) and (2) for the field fluctuation and the stress-energy tensor, respectively. Although we have constructed the general form of the stress-energy tensor up to the next-to-leading terms by functional differentiation of the action functional with respect to the metric, here we proposed a computationally simpler method in which the reduced action functionals are varied with respect to the functions $g_{tt}(r)$ and $g_{rr}(r)$. Both methods, when overlap, give, of course, identical results.

We hope that our results will be of use in further calculations. We indicate a few possible directions of investigations. First, it would be interesting to analyze the back reaction of the quantized massive field upon the geometry of the Reissner-Nordström black hole. Due to simplicity of the stress-energy tensor in the Reissner-Nordström spacetime the quantum-corrected metric can easily be constructed performing two elementary quadratures. Since the quantum part in the right hand side of the semiclassical Einstein field equations is calculated in a large mass limit it is purely geometric quantity and can be expressed solely in terms of the Riemann tensor, its covariant derivatives and contractions. This allows to treat the semiclassical theory as the higher derivative theory and construct various characteristics encoded in the geometry of the quantum-corrected black hole such as location of the horizons in nondegenerate as well as degenerate case [35, 36], equations of motion of the test particles [36, 37], temperature and entropy [37–42] and energy-momentum complexes [43]. Further, construction of the approximate stress-energy tensor as well as the field fluctuation in more complex backgrounds and the accompanying numerical calculations would certainly strengthen our understanding of the problem. Especially interesting in this regard is the problem of the lukewarm [28] and ultraextremal [44] black holes. Finally, an important and interesting continuation of the calculations presented in this paper would be construction of the next-to-leading term of the spinor and vector fields. We intend to return to this group of problems elsewhere.

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